10-2 The Complex Plane

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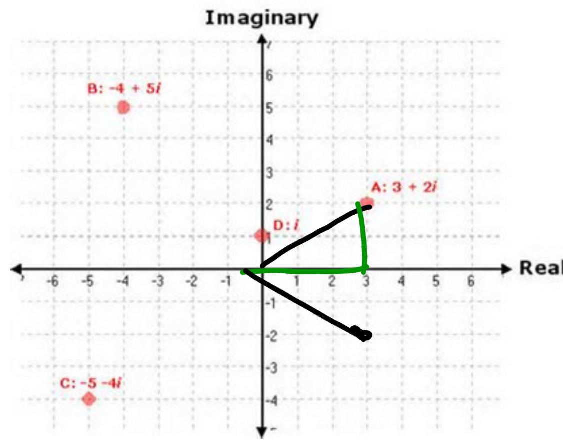
Complex numbers can be graphed in the complex plane (also called the Gauss plane or Argand plane).

The length (also called the magnitude, modulus, or absolute value) of a complex number is the distance it is away from the origin and is given by:

$$|z| = \sqrt{x^2 + y^2}$$



And it follows immediately that the conjugate would have the same magnitude. $|z^*| = |z|$

$z_1 = 4 - 5i$

$z_2 = 2 + 3i$

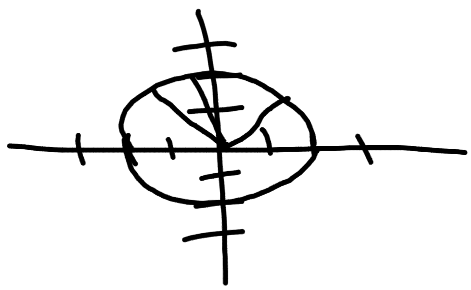
Find: $z_1 + z_2$ $6-2i$

$z_1 - z_2$ $2-8i$

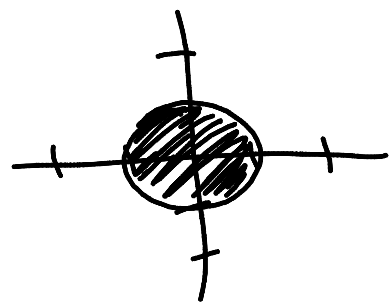
and show results graphically.

Ex1. Graph each set of complex numbers.

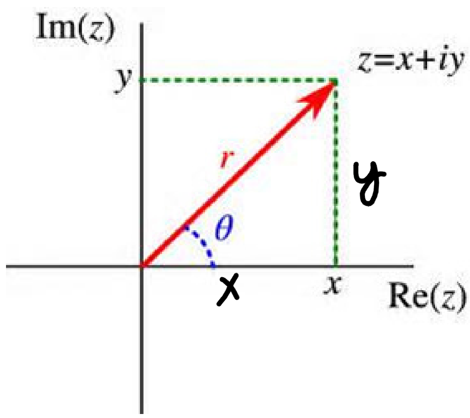
1. $A = \{z / |z| = 2\}$



2. $B = \{z / |z| \leq 1\}$



Polar Form of a Complex Number



$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$r = \sqrt{x^2 + y^2}$$

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$\underset{\text{C.F.}}{x + yi} = \underbrace{r\cos\theta + ir\sin\theta}_{\text{P.F.}}$$

The angle θ is called the argument of the complex number and the length r = |z| is called the modulus.

$$x + yi = r\cos\theta + ir\sin\theta$$

This form is called the “modulus-argument” form of a complex number (by IB).

$$z = 3 + 4i$$

$$r = \sqrt{3^2 + 4^2} = 5$$

$$\theta = \tan^{-1}\left(\frac{4}{3}\right) \approx 53.1^\circ$$

$$z = 5\left(\cos 53.1^\circ + i\sin 53.1^\circ\right)$$

$$z = 5cis53.1^\circ$$

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Ex2. Write in modulus-argument form

$$z = 3 + i\sqrt{3}$$

$r = \sqrt{3^2 + (\sqrt{3})^2} = \sqrt{12}$

$\theta = \tan^{-1}\left(\frac{\sqrt{3}}{3}\right) = 30^\circ$

$z = \sqrt{12}\,\text{cis}\,30^\circ$

Multiplying Complex Numbers in Modulus-Argument Form

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$$
$$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$
$$z_1 \cdot z_2 = r_1(\cos\theta_1 + i\sin\theta_1)\cdot r_2(\cos\theta_2 + i\sin\theta_2)$$
$$= r_1 \cdot r_2 \cdot (\cos\theta_1 + i\sin\theta_1)\cdot(\cos\theta_2 + i\sin\theta_2)$$
$$= r_1 \cdot r_2 \cdot (\cos\theta_1 \cdot \cos\theta_2 + \cos\theta_1 \cdot i\sin\theta_2 + i\sin\theta_1 \cdot \cos\theta_2 + i^2 \sin\theta_1 \cdot \sin\theta_2)$$
$$= r_1 \cdot r_2 \cdot (\cos\theta_1 \cdot \cos\theta_2 + \cos\theta_1 \cdot i\sin\theta_2 + i\sin\theta_1 \cdot \cos\theta_2 - \sin\theta_1 \cdot \sin\theta_2)$$
$$= r_1 \cdot r_2 \cdot \left[(\cos\theta_1 \cdot \cos\theta_2 - \sin\theta_1 \cdot \sin\theta_2) + i(\cos\theta_1 \cdot \sin\theta_2 + \sin\theta_1 \cdot \cos\theta_2)\right]$$
$$= r_1 \cdot r_2 \cdot \left[(\cos(\theta_1 + \theta_2) + i(\sin(\theta_1 + \theta_2))\right]$$
$$= r_1 \cdot r_2 \cdot cis(\theta_1 + \theta_2)$$

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When **multiplying complex numbers in polar form**, multiply the modului and add the arguments.

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$$
$$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

$$z_1 \cdot z_2 = r_1 \cdot r_2 \cdot \left[(\cos(\theta_1+\theta_2) + i(\sin(\theta_1+\theta_2))\right]$$
$$z_1 \cdot z_2 = r_1 \cdot r_2 \cdot cis(\theta_1+\theta_2)$$

Ex3. Mulitply

$$z_1 = \sqrt{2}cis\left(\frac{\pi}{4}\right)$$

$$z_2 = \sqrt{5}cis\left(\frac{\pi}{3}\right)$$

$$\sqrt{10}cis\left(\frac{7\pi}{12}\right)$$

$$\sqrt{10}\cos\frac{7\pi}{12} + \sqrt{10}\, i \sin\frac{7\pi}{12}$$

When **dividing complex numbers in polar form**, divide the modului and subtract the arguments.

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$$
$$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\left[(\cos(\theta_1 - \theta_2) + i(\sin\theta_1 - \theta_2)\right]$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} cis(\theta_1 - \theta_2)$$

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Ex4. Divide $\dfrac{z_1}{z_2}$

$z_1 = \sqrt{2} cis\left(\dfrac{\pi}{4}\right)$

$z_2 = \sqrt{5} cis\left(\dfrac{\pi}{3}\right)$

$$\frac{\sqrt{2}}{\sqrt{5}} cis\left(\frac{-\pi}{12}\right)$$

$$\frac{\sqrt{10}}{5} cis\left(\frac{-\pi}{12}\right)$$

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Ex5a. Find the exact value

$\sin\left(\frac{7\pi}{12}\right)$

$\sin\left(\frac{\pi}{4}+\frac{\pi}{3}\right)$

$\sin\frac{\pi}{4}\cos\frac{\pi}{3}+\cos\frac{\pi}{4}\sin\pi/3$

$\left(\frac{\sqrt{2}}{2}\cdot\frac{1}{2}\right)+\left(\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2}\right)$

$\frac{\sqrt{2}}{4}+\frac{\sqrt{6}}{4}$

$\frac{\sqrt{2}+\sqrt{6}}{4}$

$\cos\left(\frac{7\pi}{12}\right)$

$\cos\left(\frac{\pi}{4}+\frac{\pi}{3}\right)$

$\cos\frac{\pi}{4}\cos\frac{\pi}{3}$

$-\sin\frac{\pi}{4}\sin\frac{\pi}{3}$

$\frac{\sqrt{2}}{2}\cdot\frac{1}{2}-\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2}$

$\frac{\sqrt{2}-\sqrt{6}}{4}$

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Ex5b. Hence, write $z_1 \cdot z_2$ in component form with exact values.

$$\sqrt{10}\ \text{cis}\left(\frac{7\pi}{12}\right)$$

$$\sqrt{10}\left(\frac{\sqrt{2}-\sqrt{6}}{4} + i\,\frac{\sqrt{2}+\sqrt{6}}{4}\right)$$

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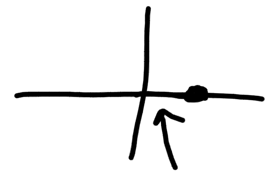
Ex5. $z = \sqrt{2}cis\left(\frac{\pi}{4}\right)$ find: $\frac{1}{z}$



$1 = 1 cis 0$

$$\frac{1}{\sqrt{2}\ cis\ \pi/4}$$

$$\frac{1\ cis\ 0}{\sqrt{2}\ cis\ \pi/4}$$

$$\frac{1}{\sqrt{2}} cis\left(\frac{-\pi}{4}\right)$$

$$\frac{\sqrt{2}}{2} cis\left(\frac{-\pi}{4}\right)$$

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